COUNTABLY HYPERCYCLIC OPERATORS

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Abstract. Motivated by Herrero's conjecture on finitely hypercyclic operators, we define countably hypercyclic operators and establish a Countably Hypercyclic Criterion that is surprisingly similar to the well known Hypercyclicity Criterion. Our results support the idea that there is a countable version of Herrero's Conjecture for invertible operators.

We use our criterion to characterize the hyponormal operators whose adjoints are countably hypercyclic and to give examples of countably hypercyclic operators that are not cyclic.

1. Introduction

A bounded linear operator $T : X \to X$ on a separable Banach space X is *hypercyclic* if it has a vector with dense orbit. That is, if there exists a vector $x \in X$ such that its orbit $\{T^n x : n \geq 0\}$ is dense in X. We shall denote the orbit of x under T by $\mathrm{Orb}(T, x)$. Examples of hypercyclic operators abound, they arise in the classes of backward weighted shifts [17], adjoints of multiplication operators on spaces of analytic functions [11], composition operators [3], and adjoints of hyponormal operators [8]. In 1992, Herrero [12] made the conjecture that a finitely hypercyclic operator is necessarily hypercyclic. That is, if there exists a finite set of vectors, say $\{x_1, \dots, x_n\}$, such that $\bigcup_{k=1}^{n} \mathrm{Orb}(T, x_k)$ is dense, then T must be hypercyclic. That this is so, was proven independently by Costakis [6] and Peris [16], also see Bourdon and Feldman [4].

A natural question, raised by V. Miller [15] and (privately) by W. Wogen asks if there is a version of Herrero's conjecture involving countably many vectors.

A sequence of vectors $(x_k)_{k=1}^{\infty}$ is separated if there exists an $\epsilon > 0$ such that $\|x_n - x_k\| \geq \epsilon$ for all $n \neq k$. We shall call an operator *countably hypercyclic* if there is a bounded separated sequence $(x_k)_{k=1}^{\infty}$ such that $\bigcup_{k=1}^{\infty} \mathrm{Orb}(T, x_k)$ is dense. Our results support the following question:

Question 1.1. *If T is invertible and both T and T^{-1} are countably hypercyclic, then T must be hypercyclic.*

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If an operator T is invertible and finitely hypercyclic, then one can easily show that T^{-1} is also finitely hypercyclic. Thus our conjecture is a natural extension of Herrero's finitely hypercyclic conjecture, in the invertible case.

However, we also shall show that there are (invertible) countably hypercyclic operators on Hilbert space that are not cyclic. Nevertheless, there is strong evidence that if both T and T^{-1} are countably hypercyclic, then T should be hypercyclic. Also, it is surprising how many operators have the property that they are countably hypercyclic. We shall present a "Countably Hypercyclic Criterion" very similar to the well known Hypercyclicity Criterion due to Kitai, Gethner and Shapiro, ([13] and [10]).

We use our criterion to give a local spectral theory condition for an operator to be countably hypercyclic and use this to characterize the hyponormal operators with countably hypercyclic adjoints. We also prove that hyponormal operators cannot be countably hypercyclic, and that a backward weighted shift is countably hypercyclic if and only if it is hypercyclic.

We also address the more general question of when an operator has a bounded set with dense orbit.

In what follows X will denote a separable infinite dimensional complex Banach space, $\mathcal{B}(X)$ will denote the algebra of all bounded linear operators on X, $\mathbb{D}$ will denote the open unit disk in the complex plane $\mathbb{C}$, and for $a \in X$ and $r > 0$, $B(a, r)$ will denote the open ball at a with radius r.

2. Two examples of what can go wrong

If C is any subset of X then the orbit of C under T is $\bigcup\{T^n(C) : n \geq 0\}$, and shall be denoted by $\text{Orb}(T, C)$. Notice that $\text{Orb}(T, C) = \bigcup_{x \in C} \text{Orb}(T, x)$.

The following two examples give different situations where an operator has a countable set with dense orbit. Notice that in both examples one may choose T to be a multiple of the identity operator! These examples suggest that a suitable definition of countable hypercyclicity should require the countable set to be bounded, bounded away from zero, and have no cluster points.

Example 2.1. *If $T \in \mathcal{B}(X)$ and $\sigma(T) \subseteq (\mathbb{C} \setminus \overline{\mathbb{D}})$, then there is a sequence $(x_n)_{n=1}^{\infty} \subseteq X$ such that $\|x_n\| \to 0$ and $\bigcup_{n=1}^{\infty} Orb(T, x_n)$ is dense in X.*

Proof. Suppose that $(y_n)_{n=1}^{\infty}$ is a countable dense set in $B(0, 1) = \{x \in X : \|x\| < 1\}$. Let $x_n = T^{-n} y_n$, since $\sigma(T^{-1}) \subseteq \mathbb{D}$, it follows from the spectral radius formula that $\|T^{-n}\| \to 0$, thus $\|x_n\| \to 0$. If $C = (x_n)_{n=1}^{\infty}$, then one easily checks that $y_n \in \text{Orb}(T, C)$, thus $B(0, 1) \subset \text{clOrb}(T, C)$. Now if $x \in X$, then $T^{-n} x \to 0$, so for all large n, $T^{-n} x \in B(0, 1) \subseteq \text{clOrb}(T, C)$. But since $\text{clOrb}(T, C)$ is invariant under T it follows easily that $x \in \text{clOrb}(T, C)$. Thus $\text{clOrb}(T, C) = X$. □

Example 2.2. *If $T \in \mathcal{B}(X)$ is invertible and $\|T\| < 1$, then there is a sequence $(x_n)_{n=1}^{\infty} \subseteq X$ such that $\|x_n\| \to \infty$ and $\bigcup_{n=1}^{\infty} Orb(T, x_n)$ is dense in X.*

Proof. Let $(y_n)_{n=1}^{\infty}$ be a countable dense set in $V = \{x \in X : \|x\| > 1\}$. Now let $x_n = T^{-n} y_n$. Then $\|x_n\| \geq \frac{1}{\|T\|^n} \to \infty$. If $C = (x_n)_{n=1}^{\infty}$, then one easily checks that $y_n \in \mathrm{Orb}(T, C)$, thus $V \subset \mathrm{clOrb}(T, C)$. Now if $x \in X$ and $x \neq 0$, then $\|T^{-n}x\| \to \infty$, so for all large n, $T^{-n}x \in V \subseteq \mathrm{clOrb}(T, C)$. But since $\mathrm{clOrb}(T, C)$ is invariant under T it follows easily that $x \in \mathrm{clOrb}(T, C)$. Thus $\mathrm{clOrb}(T, C) = X$. □

3. The Countably Hypercyclic Criterion

In this section we present a criterion for an operator to be countably hypercyclic. Our criterion is surprisingly close to the well known Hypercyclicity Criterion, it requires only a slight weakening of the hypothesis of the Hypercyclicity Criterion.

For comparison purposes we include the Hypercyclicity Criterion that is due independently to C. Kitai [13] and Gethner and Shapiro [10]. The Criterion has been used to show that a wide variety of linear operators are hypercyclic, and it has been asked whether or not a slight generalization of it is both necessary and sufficient for an operator to be hypercyclic, see [1].

Theorem 3.1 (The Hypercyclicity Criterion)**.** *Suppose that $T \in \mathcal{B}(X)$. If there exists two dense subspaces Y and Z in X such that:*

1. $T^n x \to 0$ *for every* $x \in Y$, *and*
2. *There exists functions* $B_n : Z \to X$ *such that* $T^n B_n = I|Z$ *and* $B_n x \to 0$ *for every* $x \in Z$,

then T is hypercyclic.

The Hypercyclicity Criterion often only requires Y and Z to be dense subsets, however by taking their linear spans, there is no loss of generality in assuming that they are subspaces (and B_n is linear, but not necessarily continuous). What the Hypercyclicity Criterion says is that if T has a dense set of vectors whose orbits go to zero and a dense set of vectors which have backward orbits that go to zero, then T is hypercyclic.

Theorem 3.2 (The Countably Hypercyclic Criterion)**.** *Suppose that $T \in \mathcal{B}(X)$. If there exists two subspaces Y and Z in X, where Y is infinite dimensional and Z is dense in X such that:*

1. $T^n x \to 0$ *for every* $x \in Y$, *and*
2. *There exists functions* $B_n : Z \to X$ *such that* $T^n B_n = I|Z$ *and* $B_n x \to 0$ *for every* $x \in Z$,

then T is countably hypercyclic.

Remark. One may replace conditions (1) and (2) in the above criterion with the assumptions that $T^{n_k} x \to 0$ and $T^{n_k} B_{n_k} x \to x$, and $B_{n_k} x \to 0$ for some subsequence of integers $n_k \to \infty$ and obtain a somewhat more general criterion, as was done with the Hypercyclicity Criterion in [1].

We'll say that a sequence $(x_n)_{n=1}^\infty$ is ϵ-separated if $\|x_n - x_k\| \geq \epsilon$ for all $n \neq k$. Then a sequence $(x_n)_{n=1}^\infty$ is separated if it is ϵ-separated from some $\epsilon > 0$.

Lemma 3.3. *If Y is an infinite dimensional subspace (not necessarily closed) of a Banach space X, then there exists a sequence $(x_n)_{n=1}^\infty \subseteq Y$ such that $\|x_n\| = 1$ for all $n \geq 1$ and $\|x_n - x_k\| > 1$ for all $n \neq k$.*

The previous lemma is an easy exercise using the Hahn-Banach Theorem (see [7, page 7]). In a Hilbert space it simply reduces to a Gram-Schmidt argument, since any orthonormal sequence (x_n) satisfies $\|x_n - x_m\| = \sqrt{2}$ for $n \neq m$. It is a deep theorem due to J. Elton and E. Odell that for a given normed linear space, there is an $\epsilon > 0$ and a sequence of unit vectors that are $(1+\epsilon)$-separated [7, page 241].

Proof of Theorem 3.2. Let $(z_n)_{n=1}^\infty$ be a countable dense subset of Z and by Lemma 3.3, let $(y_n)_{n=1}^\infty$ be a 1-separated sequence of unit vectors in Y. Fix a $0 < \delta < 1$. For each n, let k_n be chosen such that $\|B_{k_n} z_n\| \leq \frac{\delta}{2n}$ and $\|T^{k_n} y_n\| \leq \frac{\delta}{2n}$. Now let $x_n = y_n + B_{k_n} z_n$. Clearly, $C = (x_n)_{n=1}^\infty$ is a bounded $(1-\delta)$-separated sequence. To show that $\text{Orb}(T, C)$ is dense in X, suppose that $x \in X$ and let $\epsilon > 0$. Choose an n large enough such that $\frac{1}{n} < \epsilon$ and such that $\|z_n - x\| < \epsilon/2$. Then since $T^{k_n} x_n = T^{k_n} y_n + z_n$, it follows that $\|x - T^{k_n} x_n\| \leq \|x - z_n\| + \|T^{k_n} y_n\| < \epsilon$. Thus $\text{Orb}(T, C)$ is dense in X. □

4. Local Spectral Theory and Cohyponormal Operators

We will now give a local spectral theory condition on an operator that guarantees it will satisfy the Countably Hypercyclic Criterion. For an introduction to local spectral theory, see [14].

Suppose that $T \in \mathcal{B}(X)$. If $K \subseteq \mathbb{C}$ is a closed set, then define $X_T(K) = \{x \in X :$ there exists an analytic function $f : (\mathbb{C} \setminus K) \to X$ satisfying $(T - \lambda) f(\lambda) = x$ for all $\lambda \in (\mathbb{C} \setminus K)\}$. If $U \subseteq \mathbb{C}$ is an open set, then define $X_T(U) = \bigcup\{X_T(K) : K \subseteq U$ is compact$\}$. One easily checks that $X_T(U)$ contains all eigenvectors for T whose eigenvalues belong to U and that $X_T(U)$ is a hyperinvariant subspace for T, although it is not necessarily closed.

The following Theorem appears in Feldman, Miller and Miller [8]

Theorem 4.1. *Suppose that $T \in \mathcal{B}(X)$. If $X_T(\mathbb{D})$ and $X_T(\mathbb{C} \setminus \overline{\mathbb{D}})$ is dense, then T is hypercyclic.*

Proof. It is shown in [8] that if we set $Y = X_T(\mathbb{D})$ and $Z = X_T(\mathbb{C} \setminus \overline{\mathbb{D}})$, then the conditions of the Hypercyclicity Criterion are satisfied. □

Essentially, in [8], it is shown that vectors in $X_T(\mathbb{D})$ have orbits that go to zero and vectors in $X_T(\mathbb{C} \setminus \overline{\mathbb{D}})$ have backward orbits that go to zero. Next we present the countably hypercyclic analogue of Theorem 4.1.

Theorem 4.2. *Suppose that $T \in \mathcal{B}(X)$. If $X_T(\mathbb{D})$ is infinite dimensional and $X_T(\mathbb{C} \setminus \overline{\mathbb{D}})$ is dense, then T is countably hypercyclic.*

Proof. Simply let $Y = X_T(\mathbb{D})$ and $Z = X_T(\mathbb{C} \setminus \overline{\mathbb{D}})$, then one can show that T satisfies the Countably Hypercyclic Criterion, see Feldman, Miller and Miller [8, Theorem 3.2] for the computation. □

Corollary 4.3. *Suppose that $T \in \mathcal{B}(X)$. If span$\{ker(T-\lambda) : |\lambda| < 1\}$ is infinite dimensional and span$\{ker(T-\lambda) : |\lambda| > 1\}$ is dense in X, then T is countably hypercyclic.*

The following proposition is straightforward and similar to analogous results for hypercyclic operators.

Proposition 4.4. *(a) If $T \in \mathcal{B}(X)$ and there is a bounded set C with $Orb(T, C)$ dense, then*

1. *$sup\|T^n\| = \infty$,*
2. *$\sigma_p(T^*) \cap \overline{\mathbb{D}} = \emptyset$, in particular T has dense range.*
3. *Every quotient of T by an invariant subspace must have norm greater than one.*
4. *No component of $\sigma(T)$ can be contained in the open unit disk.*
5. *T^* cannot have a non-zero bounded orbit.*

(b) If there is a set C that is bounded away from zero and $Orb(T, C)$ is dense, then T cannot be expansive, that is there exists an $x \in X$ such that $\|Tx\| < \|x\|$.

Proof. We shall prove (5). Suppose that $y \in X^*$ has a bounded orbit under T^* and $y \neq 0$. Let $M = \sup\|T^{*n}y\|$ and let $N = \sup\{\|z\| : z \in C\}$. Since $\{T^n x : x \in C, n \geq 0\}$ is dense in X, and since $y \neq 0$ it follows that $\{\langle T^n x, y\rangle : x \in C, n \geq 0\}$ is dense in $\mathbb{C}$ (where we use $\langle x, y\rangle$ to denote the value of the linear functional y at the vector x). However notice that for $x \in C$, $|\langle T^n x, y\rangle| = |\langle x, T^{*n}y\rangle| \leq \|x\|\|T^{*n}y\| \leq NM$. Thus, $\{\langle T^n x, y\rangle : x \in C, n \geq 0\}$ is bounded, a contradiction to the fact that it is also dense. Thus, T^* has no non-zero bounded orbits. □

Remark. (1) Notice that if T is countably hypercyclic and $C = (x_n)$ is a bounded separated sequence with dense orbit, then one may assume that $x_n \neq 0$ for all n, thus it follows that C is both bounded and bounded away from zero.

(2) The condition $\sup\|T^n\| = \infty$ on an operator means (by the principle of uniform boundedness) that T has a vector with an unbounded orbit, which is easily seen to be equivalent to saying that T has *sensitive dependence on initial conditions*, in the sense of Devaney [5].

Lemma 4.5. *(a) If S is a hyponormal operator on a Hilbert space $\mathcal{H}$, then for any open set $U \subseteq \mathbb{C}$, we have $\mathcal{H}_{S^*}(U)^{\perp} = \mathcal{H}_S(\mathbb{C} \setminus U)$.*

(b) If S is a pure hyponormal operator for which $\mathcal{H}_{S^*}(\mathbb{D})$ is finite dimensional, then $\mathcal{H}_{S^*}(\mathbb{D}) = \{0\}$.

Proof. (a) follows from [14, Proposition 2.5.14].

(b) Suppose that $\mathcal{H}_{S^*}(\mathbb{D})$ is non-zero and finite dimensional. Since $\mathcal{H}_{S^*}(\mathbb{D})$ is a finite dimensional invariant subspace for S^*, it follows that S^* has eigenvectors with eigenvalues in $\mathbb{D}$. Let λ be such an eigenvalue, then since $\ker(S^* - \lambda) \subseteq \mathcal{H}_{S^*}(\mathbb{D})$, it follows that $\ker(S^* - \lambda)$ is finite dimensional. Thus, $(S - \overline{\lambda})$ has closed range. Thus since S is pure, $(S - \overline{\lambda})$ is one-to-one with closed range, hence $\overline{\lambda} \in [\sigma(S) \setminus \sigma_{ap}(S)]$. However, $[\sigma(S) \setminus \sigma_{ap}(S)]$ is an open set and since $\lambda \in \mathbb{D} \cap [\sigma(S) \setminus \sigma_{ap}(S)]$, it follows that $\mathbb{D} \cap [\sigma(S) \setminus \sigma_{ap}(S)]$ is a non-empty open set. Hence for each $\mu \in \mathbb{D} \cap [\sigma(S) \setminus \sigma_{ap}(S)]$ we have $\ker(S^* - \overline{\mu}) \neq (0)$ and $\ker(S^* - \overline{\mu}) \subseteq \mathcal{H}_{S^*}(\mathbb{D})$. It follows that $\mathcal{H}_{S^*}(\mathbb{D})$ is infinite dimensional, a contradiction. □

Theorem 4.6. *Suppose that S is a pure hyponormal operator on a separable Hilbert space $\mathcal{H}$, then S^* is countably hypercyclic if and only if for every hyperinvariant subspace $\mathcal{M}$ of S, $\sigma(S|\mathcal{M}) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$ and $\sigma(S) \cap \mathbb{D} \neq \emptyset$.*

Proof. Suppose the spectral conditions are satisfied. We want to apply Theorem 4.2. So, suppose that $\mathcal{H}_{S^*}(\mathbb{D}) = (0)$. Since $\mathcal{H}_{S^*}(\mathbb{D})^{\perp} = \mathcal{H}_S(\mathbb{C} \setminus \mathbb{D})$, it follows that $\mathcal{H}_S(\mathbb{C} \setminus \mathbb{D}) = \mathcal{H}$. Thus, $\sigma(S) = \sigma(S|\mathcal{H}_S(\mathbb{C} \setminus \mathbb{D})) \subseteq (\mathbb{C} \setminus \mathbb{D})$, a contradiction. So, $\mathcal{H}_{S^*}(\mathbb{D}) \neq (0)$, now by Lemma 4.5, it follows that $\mathcal{H}_{S^*}(\mathbb{D})$ is infinite dimensional. Now, suppose that $\mathcal{H}_{S^*}(\mathbb{C} \setminus \overline{\mathbb{D}})$ is not dense in $\mathcal{H}$. Then, by Lemma 4.5, $\mathcal{H}_S(\overline{\mathbb{D}})$ is a non-zero hyperinvariant subspace for S. Furthermore, $\sigma(S|\mathcal{H}_S(\overline{\mathbb{D}})) \subseteq \overline{\mathbb{D}}$, contradicting our assumption. Thus it follows that $\mathcal{H}_{S^*}(\mathbb{C} \setminus \overline{\mathbb{D}})$ is dense. So, by Theorem 4.2, S^* is countably hypercyclic.

Conversely, suppose S^* is countably hypercyclic. Let C be a bounded set, that is bounded away from zero, with dense orbit. Let $\mathcal{M}$ be an invariant subspace for S and let P be the projection onto $\mathcal{M}$. Since $P : \mathcal{H} \to \mathcal{M}$ intertwines S^* and $(S|\mathcal{M})^*$, it follows that $P(C)$ is *bounded* set whose orbit under $(S|\mathcal{M})^*$ is dense in $\mathcal{M}$. Thus, we must have $\|(S|\mathcal{M})\| = \|(S|\mathcal{M})^*\| > 1$. Since the spectral radius of $(S|\mathcal{M})$ equals its norm, it follows that $\sigma(S|\mathcal{M}) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$.

Now, if $\sigma(S) \cap \mathbb{D} = \emptyset$, then $\|(S^*)^{-1}\| = \|S^{-1}\| \leq 1$, thus $\|S^* x\| \geq \|x\|$ for all $x \in \mathcal{H}$, contradicting Proposition 4.4(b). □

The next corollary follows immediately from the proof above. It says that if S is a pure hyponormal operator and S^* has a bounded set C with dense orbit and C is also bounded away from zero, then, in fact, we can find a bounded separated sequence with dense orbit.

Corollary 4.7. *If S is a pure hyponormal operator, then S^* is countably hypercyclic if and only if S^* has a bounded set, which is also bounded away from zero, that has dense orbit.*

Notice that if an operator T has a set with dense orbit, then any non-zero multiple of that set also has dense orbit. Thus T has a bounded set with dense orbit if and only if the unit ball has dense orbit if and only if $B(0, r)$ has dense orbit for any $r > 0$.

Corollary 4.8. *If S is a hyponormal operator, then S^* has a bounded set with dense orbit if and only if for every hyperinvariant subspace $\mathcal{M}$ of S, $\sigma(S|\mathcal{M}) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$.*

Proof. As in the proof of Theorem 4.6, if the spectral condition is satisfied, then $\mathcal{H}_{S^*}(\mathbb{C} \setminus \overline{\mathbb{D}})$ is dense in $\mathcal{H}$. It follows that if $Z = \mathcal{H}_{S^*}(\mathbb{C} \setminus \overline{\mathbb{D}})$, then condition (2) of the Countably Hypercyclic Criterion is satisfied, see Feldman, Miller and Miller [8, Theorem 3.2]. However, condition (2) of the Countably Hypercyclic Criterion easily implies that the unit ball has dense orbit. The converse is similar to the proof of Theorem 4.6. □

For comparison purposes we state the following result due to Feldman, Miller and Miller [8, Theorem 4.3].

Theorem 4.9. *Suppose that S is a hyponormal operator on a separable Hilbert space $\mathcal{H}$, then S^* is hypercyclic if and only if for every hyperinvariant subspace $\mathcal{M}$ of S, $\sigma(S|\mathcal{M}) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$ and $\sigma(S|\mathcal{M}) \cap \mathbb{D} \neq \emptyset$.*

Corollary 4.10. *If S is an invertible hyponormal operator and S^* and $(S^*)^{-1}$ are both countably hypercyclic, then S^* is hypercyclic. In fact, if S^* and $(S^*)^{-1}$ both have bounded sets with dense orbit, then S^* is hypercyclic.*

The above corollary follows easily from Theorem 4.6 and Theorem 4.9.

Example 4.11. *Let $S_n = M_z$ on the Bergman space, $L^2_a(\Delta_n)$, where $\{\Delta_n : 1 \leq n < \infty\}$ is a bounded collection of open disks. Let $S = \bigoplus_{n=1}^{\infty} S_n$. Then:*

(a) S^* has a bounded set with dense orbit if and only if every disk Δ_n intersects $\{z : |z| > 1\}$.

(b) S^* is countably hypercyclic if and only if every disk Δ_n intersects $\{z : |z| > 1\}$ and at least one disk intersects $\{z : |z| < 1\}$.

(c) S^* is hypercyclic if and only if every disk Δ_n intersects both $\{z : |z| < 1\}$ and $\{z : |z| > 1\}$

Proof. Clearly, S is a pure subnormal operator. Simply observe that any hyperinvariant subspace $\mathcal{M}$ of S must have the form $\mathcal{M} = \bigoplus_n \mathcal{M}_n$ where each $\mathcal{M}_n$ is a hyperinvariant subspace of S_n. Now, if $\mathcal{M} \neq (0)$, then $\mathcal{M}_n \neq (0)$ for some n. However one easily sees that $\sigma(S_n|\mathcal{M}_n) = \mathrm{cl}\Delta_n$. Thus, $\mathrm{cl}\Delta_n \subseteq \sigma(S_n|\mathcal{M}_n) \subseteq \sigma(S|\mathcal{M})$.

Also, for a given integer k, if $\mathcal{M} = \bigoplus_n \mathcal{M}_n$ where $\mathcal{M}_k = L^2_a(\Delta_k)$ and $\mathcal{M}_n = (0)$ for $n \neq k$, then $\sigma(S|\mathcal{M}) = \mathrm{cl}\Delta_k$.

With these observations, one may apply the above theorems to obtain the result. □

5. Bounded Sets with Dense Orbit

Here we consider the weaker condition of having a bounded set with dense orbit, or having a bounded set that is bounded away from zero with dense orbit. We will show that hyponormal operators cannot have the latter property and that a backward weighted shift with the former property is actually hypercyclic. We also characterize the subnormal operators that have bounded sets with dense orbit as those whose spectral measures are carried on $\{z \in \mathbb{C} : |z| > 1\}$.

Observe that an operator T will have a bounded set with dense orbit exactly when the unit ball has dense orbit. Furthermore, the following proposition shows that roughly speaking the unit ball has dense orbit under T exactly when T has a dense set of vectors with backward orbits that cluster at zero. We'll state this for invertible operators since a sharper, cleaner result can then be obtained.

Proposition 5.1. *If $T : X \to X$ is invertible, then the open unit ball has dense orbit if and only if there is a dense G_δ set $Y \subseteq X$ such that for every $y \in Y$, $\liminf_n \|T^{-n}y\| = 0$.*

Proof. Suppose that the unit ball has a dense orbit under T. Since any non-zero multiple of a set with dense orbit also has dense orbit, it follows that $B(0, r)$ has dense orbit for every $r > 0$. So for each $n \in \mathbb{N}$, $\mathrm{Orb}(T, B(0, 1/n))$ is a dense open set. Thus $Y = \bigcap_{n\in\mathbb{N}} \mathrm{Orb}(T, B(0, 1/n))$ is a dense G_δ set with the required property. □

The next proposition tells exactly when a subnormal operator has a bounded set with dense orbit, it should be contrasted with Corollary 4.8 that describes when a cohyponormal operator has a bounded set with dense orbit. Also see Example 4.11 for an example with Bergman operators.

Proposition 5.2. *(a) If $T \in \mathcal{B}(X)$ and $\sigma(T) \subseteq \{z : |z| > 1\}$, then $Orb(T, B(0, 1))$ is dense in X.*

(b) If S is a subnormal operator on a Hilbert space, then $Orb(S, B(0, 1))$ is dense if and only if the spectral measure for S is carried by $\{z : |z| > 1\}$.

Proof. (a) If $\sigma(T) \subseteq \{z : |z| > 1\}$, then T is invertible and $\|T^{-n}x\| \to 0$ for every x, thus T for all large n, $x \in T^n(B(0, 1))$. So, $B(0, 1)$ has dense orbit.

(b) If S is subnormal and its spectral measure is carried by $\{z : |z| > 1\}$, then for every x we have that $\|S^{-n}x\| \to 0$, thus the unit ball has dense orbit. Conversely, suppose that the unit ball has dense orbit. Then by Proposition 4.4, S^* cannot have a bounded orbit. However if the spectral measure places positive mass on $\{z : |z| \le 1\}$, then one can easily show that S^* does have a bounded orbit. □

Proposition 5.3. *If T is a backward unilateral weighted shift (with positive weights), then T is countably hypercyclic if and only if T is hypercyclic. In fact, if T has a bounded set with dense orbit, then T is hypercyclic.*

Proof. Suppose that $Te_j = w_{j-1}e_{j-1}$ where w_j are the positive weights for T and $(e_j)_{j=0}^{\infty}$ is the canonical basis for $\ell^2(\mathbb{N})$. It was proven by Salas [17] that T is hypercyclic if and only if $sup_n \prod_{i=0}^{n} w_i = \infty$. We shall verify this condition. Suppose that C is a bounded set with dense orbit. Then by Proposition 4.4, T^* cannot have a bounded orbit. So, consider orbit of e_0 under T^*. Since T^* is a forward unilateral weighted shift, it follows that $T^{*n}e_0 = (\prod_{i=0}^{n-1} w_i)e_n$. Since this orbit cannot be bounded, it follows that Salas' condition is satisfied. Thus T is hypercyclic. □

In [17], H. Salas characterized the bilateral weighted shifts that are hypercyclic. In [9] Feldman gives a simplier criterion for *invertible* weighted shifts, that is still necessary and sufficient for hypercyclitiy (and supercyclicity).

Theorem 5.4. *If $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is an invertible bilateral weighted shift with weight sequence $(w_n)_{n=-\infty}^{\infty}$, then T is hypercyclic if and only if there exists a sequence of integers $n_k \to \infty$ such that*

$$\lim_{k\to\infty} \prod_{j=1}^{n_k} w_j = 0 \text{ and } \lim_{k\to\infty} \prod_{j=1}^{n_k} \frac{1}{w_{-j}} = 0$$

Proof. See Feldman [9]. □

To illustrate the difference between unilateral and bilateral shifts we give the following example that should be contrasted with Proposition 5.3.

Example 5.5. *There exists an invertible bilateral weighted shift such that both T and T^{-1} have bounded sets with dense orbit, but T is not hypercyclic.*

Proof. Suppose the weight sequence $\{w_n : n \in \mathbb{Z}\}$ is given as follows, for $n \geq 0$ the weights are $\{1, 1, \frac{1}{2}, 2, 1, 1, \frac{1}{2}, \frac{1}{2}, 2, 2, 1, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, \dots\}$ and for $n < 0$ they are $\{1, 1, 1, 2, \frac{1}{2}, 1, 1, 1, 1, 2, 2, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, 1, 1, 1, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \dots\}$. So, for each n, $w_n \in \{1, 2, \frac{1}{2}\}$. For $n \geq 0$, the weights consist of a string of one's, then a string of $\frac{1}{2}$'s, then a string of two's, then starting over with a string of one's. For $n < 0$, they are one's, then two's, then $\frac{1}{2}$'s.

The idea is to choose them such that if $p_n = \prod_{j=1}^n w_j$ and $q_n = \prod_{j=1}^n \frac{1}{w_{-j}}$, then we want that $0 < p_n \leq 1$ and $0 < q_n \leq 1$ for all n and $p_n < 1$ if and only if $q_n = 1$. Given this, the condition in Theorem 5.4 will never be satisfied, hence T will not be hypercyclic. Also, however, we want a sequence $n_k \to \infty$ such that $p_{n_k} \to 0$ and a (necessarily different) sequence $m_k \to \infty$ such that $q_{m_k} \to 0$. This will mean that for every vector x with finite support $T^{n_k}x \to 0$ and $T^{-m_k}x \to 0$. Thus it follows easily that the unit ball will have dense orbit under both T and T^{-1}. □

Question 5.6. *If T is a bilateral weighted shift and T is countably hypercyclic, then must T be hypercyclic?*

Notice that if a bilateral weighted shift satisfies the Countably Hypercyclic Criterion, it must be hypercyclic. This is true because if T is a bilateral weighted shift and $T^{n_k}x \to 0$ for some non-zero x, then in fact $T^{n_k}y \to 0$ for all vectors y with finite support. Hence T would satisfy the hypercyclicity criterion.

Proposition 5.7. *Suppose that T_1 and T_2 are bounded linear operators.*

(a) If T_1 and T_2 satisfy the Countably Hypercyclic Criterion, then $T_1 \oplus T_2$ also satisfies the Countably Hypercyclic Criterion.

(b) If T_1 satisfies the Countably Hypercyclic Criterion and the spectrum of T_2 is contained in $(\mathbb{C} \setminus \overline{\mathbb{D}})$, then $T_1 \oplus T_2$ satisfies the Countably Hypercyclic Criterion.

Proof of Part (b): Let T_i act on the space X_i. Since T_1 satisfies the Countably Hypercyclic Criterion, there are subspaces Y_1 and Z_1 where Y_1 is infinite dimensional and Z_1 is dense in X_1 and for every n there exists a right inverse $\hat{B}_n$ for $T^n|Z_1$,. Now simply let $Y = Y_1 \oplus (0)$ and $Z = Z_1 \oplus X_2$ and $B_n = \hat{B}_n \oplus T_2^{-n}$. Then one easily checks that the required properties hold. The proof of (a) is similar. □

Example 5.8. *Suppose that B is the backward unilateral shift on $\ell^2(\mathbb{N})$ and $T_1 = 2B$. If T_2 is any bounded linear operator on a separable Banach space such that $\sigma(T_2) \subseteq (\mathbb{C} \setminus \overline{\mathbb{D}})$, then $T_1 \oplus T_2$ is countably hypercyclic.*

It follows from the previous example that countably hypercyclic operators need not be cyclic or even multi-cyclic, furthermore every component of their spectrum need not intersect the unit circle. However by Proposition 4.4 no component of the spectrum may lie entirely inside the open unit disk. Also the previous example shows that a direct sum of operators can be countably hypercyclic without each summand being countably hypercyclic.

The following proposition appears in Bourdon [2, prop 2.6].

Proposition 5.9. *If T is a hyponormal operator on a Hilbert space $\mathcal{H}$ and $h \in \mathcal{H}$ is such that $\|Th\| \geq \|h\|$, then $(\|T^n h\|)_{n=1}^{\infty}$ is an increasing sequence.*

It follows that orbits of a hyponormal operator can only do one of three things; either strictly decrease in norm, increase in norm, or strictly decreases in norm up to a point and increases in norm thereafter.

Theorem 5.10. *A hyponormal operator on a Hilbert space cannot be countably hypercyclic. In fact, there is no bounded set, that is also bounded away from zero with dense orbit.*

Proof. Suppose that T is a hyponormal operator on a Hilbert space $\mathcal{H}$ and C is a bounded set, that is bounded away from zero, and has dense orbit. We will show that, since C is bounded and has dense orbit it follows that for every $x \in \mathcal{H}$, $\|Tx\| \geq \|x\|$. However, then T is expansive and since C is bounded away from zero, the orbit of C could not be dense.

So suppose $x \in \mathcal{H}$ and $\|Tx\| < \|x\|$. By multiplying x by a scalar we may suppose that $\|x\| > M := \sup\{\|z\| : z \in C\}$. Since C has dense orbit, there exists a sequence $y_n \in \text{Orb}(T, C)$ such that $y_n \to x$. We may assume that $\|y_n\| > M$ for all n. However, then by Proposition 5.9 it follows that $\|Ty_n\| \geq \|y_n\|$ for each n. Hence, upon taking limits, we see that $\|Tx\| = \lim_n \|Ty_n\| \geq \lim_n \|y_n\| = \|x\|$, contradicting our assumption. □

Remark. The author would like to thank Paul Bourdon for the above proof that is simplier than the authors original one. Notice that the proof of Theorem 5.10 applies to any operator that satisfies the conclusion of Proposition 5.9 and there are non-hyponormal operators with this property.

Question 5.11. *If $T \in \mathcal{B}(X)$ has a bounded set, that is also bounded away from zero, with dense orbit, then must T be countably hypercyclic?*

As noted in Corollary 4.7, this is true for cohyponormal operators.

Question 5.12. *If T and T^{-1} are both countably hypercyclic, then must T be hypercyclic?*

Corollary 4.10 says that the previous question is true when T is a cohyponormal operator. Also, notice that if T and T^{-1} both satisfy the Countably Hypercyclic Criterion, then T satisfies the Hypercyclic Criterion. Thus the question has an affirmative answer in that case also.

Remark: Recently, Alfredo Peris has shown (private communication) that the operator T in Example 5.5 has the property that T and T^{-1} are both countably hypercyclic. This answers questions 5.6 and 5.12 negatively.

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